Final Exam — Ordinary Differential Equations (WIGDV–07)

Wednesday 1 November 2017, 14.00h–17.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (2 + 10 = 12 points)

Consider the following Riccati equation:

$$y' + (2x^3 - 1)y - x^2y^2 = x^4 - x + 1.$$

(a) Show that $\phi(x) = x$ is a solution.

(b) Compute a solution that satisfies the initial condition $y(0) = 1$.

Problem 2 (2 + 5 + 6 = 13 points)

Consider the following differential equation:

$$(x^2 - 9y^2)\,dx + 18xy\,dy = 0 \quad \text{where} \quad x > 0.$$

(a) Show that the equation is *not* exact.

(b) Compute an integrating factor of the form $M(x, y) = \phi(x)$.

(c) Compute the general solution in implicit form.

Problem 3 (4 + 12 + 4 = 20 points)

Consider the following 4×4 matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 2 \end{bmatrix}.$$

(a) Show that $\det(A - \lambda I) = (\lambda - 2)^4$.

(b) Compute the matrix J of the Jordan canonical form of A. Do *not* compute Q!

(c) Compute e^{Jt}.

Problem 4 (8 + 4 + 3 + 3 = 18 points)

Let $a > 0$ and provide the space $C([0,a]) = \{y : [0,a] \to \mathbb{R} \,:\, y \text{ is continuous}\}$ with the norm

$$\|y\| = \sup_{x\in[0,a]} |y(x)|w(x),$$

where $w : [0,a] \to \mathbb{R}$ is a strictly positive function. Consider the operator:

$$T : C([0,a]) \to C([0,a]), \quad (Ty)(x) = \int_0^x ty(t)\,dt.$$

(a) Prove that for all $y, z \in C([0,a])$ we have

$$\|Ty - Tz\| \le L\|y - z\| \quad \text{where} \quad L = \sup_{x\in[0,a]} w(x) \int_0^x \frac{t}{w(t)}dt.$$

(b) Compute the value of L for $w(x) = 1$ and $w(x) = e^{-x^2}$.

(c) Formulate Banach's fixed point theorem.

(d) Explain which of the two norms of part (b) is/are suitable for applying Banach's fixed point theorem. (It is given that with both norms $C([0,a])$ is a Banach space.)

Problem 5 (12 points)

Solve the following initial value problem:

$$4t^2u'' + 13u = 7t^2, \quad u(1) = \frac{1}{3}, \quad u'(1) = \frac{11}{3}.$$

Problem 6 (6 + 6 + 3 = 15 points)

Consider the following semi-homogeneous boundary value problem:

$$u'' + \lambda u = f(x), \quad x \in [0,1], \quad u(0) = 0, \quad u(1) = 0,$$

where $\lambda \in \mathbb{R}$ is a parameter and $f : [0,1] \to \mathbb{R}$ is a continuous function.

(a) Show that for $\lambda \le 0$ the homogeneous boundary value problem only has the solution $u = 0$.

(b) Compute for $\lambda > 0$ the Green's function $\Gamma(x, \xi; \lambda)$.

(c) Explain how the eigenvalues of the homogeneous boundary value problem can be determined from the Green's function.

End of test (90 points)

Solution of problem 1 (2 + 10 = 12 points)

(a) Setting $y = x$ gives $1 + (2x^3 - 1)x - x^4 = x^4 - x + 1$ which is indeed a correct equality. Therefore, $\phi(x) = x$ is a solution.

(2 points)

(b) Let y be a solution of the Riccati equation and consider $u = y - x$, then

$$\begin{aligned} u' &= y' - 1 \\ &= (1 - 2x^3)y + x^2y^2 + x^4 - x \\ &= (1 - 2x^3)(u + x) + x^2(u^2 + 2xu + x^2) + x^4 - x \\ &= u + x^2u^2. \end{aligned}$$

(3 points)

This is a Bernoulli equation with $\alpha = 2$. Let $z = u^{1-\alpha} = 1/u$, then

$$z' = -\frac{u'}{u^2} = -\frac{1}{u} - x^2 = -z - x^2 \quad \Leftrightarrow \quad z' + z = -x^2.$$

(3 points)

Multiplication with the integrating factor e^x gives

$$(e^x z)' = -x^2 e^x \quad \Rightarrow \quad e^x z = (-2+2x-x^2)e^x + C \quad \Rightarrow \quad z = -2+2x-x^2+Ce^{-x}.$$

(3 points)

Therefore, we get the following general solution of the Riccati equation:

$$y = u + x = \frac{1}{z} + x = x + \frac{1}{-2 + 2x - x^2 + Ce^{-x}}.$$

The initial condition $y(0) = 1$ gives $C = 3$.

(1 point)

Solution of problem 2 (2 + 5 + 6 = 13 points)

(a) Let $g = x^2 - 9y^2$ and $h = 18xy$, then $g_y = -18y$ and $h_x = 18y$. Since $g_y \neq h_x$ the differential equation is not exact.
(2 points)

(b) The function $M(x, y) = \phi(x)$ is an integrating factor if and only if

$$(g\phi)_y = (h\phi)_x \quad \Leftrightarrow \quad g_y\phi = h_x\phi + h\phi' \quad \Leftrightarrow \quad \phi' = \frac{g_y - h_x}{h}\phi \quad \Leftrightarrow \quad \phi' = -\frac{2}{x}\phi,$$

where primes denote differentiation with respect to x. An obvious solution is $\phi(x) = 1/x^2$.
(5 points)

(c) Define the function

$$F(x, y) = \int g(x, y)\phi(x)\, dx = \int 1 - \frac{9y^2}{x^2}\, dx = x + \frac{9y^2}{x} + C(y).$$

(3 points)

By construction we have that $F_x = g\phi$. Demanding that $F_y = h\phi$ gives

$$\frac{18y}{x} + C'(y) = \frac{18y}{x} \quad \Rightarrow \quad C'(y) = 0,$$

which means that we can take $C(y)$ to be a constant function. For simplicity we can choose $C(y) = 0$.
(2 points)

The general solution is now given by the implicit equation

$$x + \frac{9y^2}{x} = K,$$

where $K \in \mathbb{R}$ is an arbitrary constant.
(1 point)

Solution of problem 3 (4 + 12 + 4 = 20 points)

(a) Cleverly expanding the determinant along columns with many zeros gives:

$$\begin{aligned}
\det(A-\lambda I) &= \det\begin{bmatrix} 2-\lambda & -1 & 0 & 1 \\ 0 & 3-\lambda & -1 & 0 \\ 0 & 1 & 1-\lambda & 0 \\ 0 & -1 & 1 & 2-\lambda \end{bmatrix} && \text{(along first column)} \\
&= (2-\lambda)\det\begin{bmatrix} 3-\lambda & -1 & 0 \\ 1 & 1-\lambda & 0 \\ -1 & 1 & 2-\lambda \end{bmatrix} && \text{(along last column)} \\
&= (2-\lambda)^2 \det\begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} \\
&= (2-\lambda)^2((3-\lambda)(1-\lambda)+1) \\
&= (2-\lambda)^2(\lambda^2-4\lambda+4) \\
&= (2-\lambda)^2(\lambda-2)^2 \\
&= (\lambda-2)^2.
\end{aligned}$$

(4 points)

(b) From part (a) it follows that $\lambda = 2$ is the only eigenvalue of A. We have

$$A-\lambda I = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Counting the number of *non-pivot* columns gives

$$\dim E_\lambda^1 = \dim \operatorname{Nul}(A-\lambda I) = 2.$$

(4 points)

We have

$$(A-\lambda I)^2 = \begin{bmatrix} 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

from which we can immediately count the number of of *non-pivot* columns, which gives

$$\dim E_\lambda^2 = \dim \operatorname{Nul}(A-\lambda I)^2 = 3.$$

(3 points)

It is clear that $(A-\lambda I)^3$ is the zero matrix, and therefore

$$\dim E_\lambda^3 = \dim \operatorname{Nul}(A-\lambda I)^3 = 4.$$

(1 point)

We can now construct the *dot diagram* for A:

$$\left.\begin{aligned} r_1 &= \dim E_\lambda^1 = 2 \\ r_2 &= \dim E_\lambda^2 - \dim E_\lambda^1 = 3 - 2 = 1 \\ r_3 &= \dim E_\lambda^3 - \dim E_\lambda^2 = 4 - 3 = 1 \end{aligned}\right\} \quad \Rightarrow \quad \begin{array}{ll} \bullet & \bullet \\ \bullet & \\ \bullet & \end{array}$$

(2 points)

This means that we have a basis for the generalized eigenspaces of A consisting of 2 cycles having length 3 and 1, respectively. Therefore, J consists of a 3×3 block and a 1×1 block:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

(2 points)

(c) We can write $J = D + N$, where

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $DN = ND$ we have $e^{Jt} = e^{Dt}e^{Nt} = e^{Dt}(I + Nt + \frac{1}{2}N^2t^2)$ where we have used that $N^k = 0$ for all integers $k \geq 3$. Therefore,

$$e^{Jt} = e^{2t} \begin{bmatrix} 1 & t & \frac{1}{2}t^2 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(4 points)

Solution of problem 4 (8 + 4 + 3 + 3 = 18 points)

(a) If $y, z \in C([0,a])$ and $x \in [0,a]$, then

$$\begin{aligned}|(Ty)(x) - (Tz)(x)| &= \left| \int_0^x t(y(t) - z(t))\, dt \right| \\ &\le \int_0^x t|y(t) - z(t)|\, dt \\ &= \int_0^x |y(t) - z(t)|w(t) \cdot \frac{t}{w(t)}\, dt.\end{aligned}$$

(3 points)

Since $|y(t) - z(t)|w(t) \le \|y - z\|$ for all $0 \le t \le x \le a$ it follows that

$$|(Ty)(x) - (Tz)(x)| \le \|y - z\| \int_0^x \frac{t}{w(t)}\, dt.$$

(2 points)

Multiplying the last inequality with the function w gives

$$|(Ty)(x) - (Tz)(x)|w(x) \le \|y - z\|w(x) \int_0^x \frac{t}{w(t)}\, dt.$$

(2 points)

Since this inequality holds for all $x \in [0,a]$ we can take the supremum on both sides, which gives:

$$\|Ty - Tz\| \le L\|y - z\| \quad \text{where} \quad L = \sup_{x\in[0,a]} w(x) \int_0^x \frac{t}{w(t)} dt.$$

(1 point)

(b) For $w(x) = 1$ we obtain the value

$$L = \sup_{x\in[0,a]} \int_0^x t\, dt = \sup_{x\in[0,a]} \tfrac{1}{2}x^2 = \tfrac{1}{2}a^2.$$

(2 points)

For $w(x) = e^{-x^2}$ we obtain the value

$$L = \sup_{x\in[0,a]} e^{-x^2} \int_0^x te^{t^2}\, dt = \sup_{x\in[0,a]} e^{-x^2} \cdot \frac{e^{x^2} - 1}{2} = \sup_{x\in[0,a]} \frac{1 - e^{-x^2}}{2} = \frac{1 - e^{-a^2}}{2}.$$

(2 points)

(c) Let D be a closed, nonempty subset in a Banach space B. Let the operator $T : D \to B$ map D into itself, i.e., $T(D) \subset D$, and assume that T is a contraction: there exists a number $0 < q < 1$ such that

$$\|Tx - Ty\| \le q\|x - y\|, \qquad \forall\, x, y \in D,$$

Then the fixed point equation $Tx = x$ has precisely one solution $\bar{x} \in D$.
(3 points)

Moreover, iterations of T converge to this fixed point:

$$x_0 \in D, \quad x_{n+1} = Tx_n \quad \Rightarrow \quad \lim_{n\to\infty} x_n = \bar{x}.$$

(The last statement is not relevant to this problem.)

(d) For the application of Banach's fixed point theorem we need that $L < 1$. When $w(x) = 1$ this is only the case when $a < \sqrt{2}$. In the case $w(x) = e^{-x^2}$ we have $L < 1$ for all $a > 0$. Indeed,

$$0 < e^{-a^2} < 1 \quad \Rightarrow \quad 0 < 1 - e^{-a^2} < 1 \quad \Rightarrow \quad \frac{1 - e^{-a^2}}{2} < 1.$$

Therefore, the norm with $w(x) = e^{-x^2}$ is better suitable.
(3 points)

Solution of problem 5 (12 points)

Substituting $u = t^\lambda$ in the homogeneous differential equation gives

$$4\lambda(\lambda-1)+13=0 \quad\Leftrightarrow\quad (2\lambda-1)^2+12=0 \quad\Leftrightarrow\quad \lambda=\tfrac{1}{2}\pm\sqrt{3}i.$$

Hence, the homogeneous equation has the following general solution:

$$\begin{aligned} u &= at^{\frac{1}{2}+\sqrt{3}i} + bt^{\frac{1}{2}-\sqrt{3}i} \\ &= ae^{(\frac{1}{2}+\sqrt{3}i)\log t} + be^{(\frac{1}{2}-\sqrt{3}i)\log t} \\ &= \sqrt{t}\big[ae^{\sqrt{3}i\log t} + be^{-\sqrt{3}i\log t}\big] \\ &= \sqrt{t}\big[(a+b)\cos(\sqrt{3}\log t) + (a-b)i\sin(\sqrt{3}\log t)\big] \\ &= \sqrt{t}\big[A\cos(\sqrt{3}\log t) + B\sin(\sqrt{3}\log t)\big], \end{aligned}$$

where $A = a+b$, $B = (a-b)i$, and a and b are arbitrary complex constants.
(5 points)

As a particular solution we try $u_p = Kt^2$, where K is a constant. After substitution in the differential equation we find $K = \frac{1}{3}$. Hence, the general solution of the inhomogeneous equation is

$$u = \sqrt{t}\big[A\cos(\sqrt{3}\log t) + B\sin(\sqrt{3}\log t)\big] + \frac{t^2}{3},$$

where A and B are arbitrary constants.
(3 points)

The initial condition $u(1) = \frac{1}{3}$ gives $A + \frac{1}{3} = \frac{1}{3}$ so that $A = 0$.
(2 points)

Taking the derivative of u (and using that $A = 0$) gives

$$u' = \frac{B}{2\sqrt{t}}\sin(\sqrt{3}\log t) + B\sqrt{t}\cos(\sqrt{3}\log t)\cdot\frac{\sqrt{3}}{t} + \frac{2t}{3}.$$

The initial condition $u'(1) = \frac{11}{3}$ gives $B\sqrt{3} + \frac{2}{3} = \frac{11}{3}$ so that $B = \sqrt{3}$.
(2 points)

Solution of problem 6 (6 + 6 + 3 = 15 points)

(a) If $\lambda < 0$, then the homogeneous equation has the following general solution:

$$u(x) = ae^{-\sqrt{-\lambda}x} + be^{\sqrt{-\lambda}x},$$

where a and b are arbitrary constants. The boundary conditions imply that

$$\begin{bmatrix} 1 & 1 \\ e^{-\sqrt{-\lambda}} & e^{\sqrt{-\lambda}} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since the determinant of the coefficient matrix is nonzero it follows that $a = b = 0$. Thus, for $\lambda < 0$ the homogeneous equation only has the trivial solution. **(4 points)**

If $\lambda = 0$, then the homogeneous equation has the following general solution:

$$u(x) = a + bx,$$

where a and b are arbitrary constants. The boundary conditions imply that

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since the determinant of the coefficient matrix is nonzero it follows that $a = b = 0$. Thus, for $\lambda = 0$ the homogeneous equation only has the trivial solution. **(2 points)**

(b) If $\lambda > 0$, then the homogeneous equation has the following general solution:

$$u(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$$

(2 points)

A solution satisfying $u(0) = 0$ is given by

$$u_1(x) = \sin(\sqrt{\lambda}x),$$

and a solution satisfying $u(1) = 0$ is given by

$$u_2(x) = \cos(\sqrt{\lambda})\sin(\sqrt{\lambda}x) - \sin(\sqrt{\lambda})\cos(\sqrt{\lambda}x) = \sin(\sqrt{\lambda}(x-1)).$$

(2 points)

Their Wronskian determinant is

$$W = u_1u_2' - u_1'u_2 = \sqrt{\lambda}\sin(\sqrt{\lambda}).$$

Since $p(x) \equiv 1$ the Green's function is given by

$$\Gamma(x,\xi) = \frac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda})} \begin{cases} \sin(\sqrt{\lambda}\xi)\sin(\sqrt{\lambda}(x-1)) & \text{if } 0 \le \xi \le x \le 1, \\ \sin(\sqrt{\lambda}x)\sin(\sqrt{\lambda}(\xi-1)) & \text{if } 0 \le x \le \xi \le 1. \end{cases}$$

(2 points)

(c) The Green's function does not exist for those values of λ which are eigenvalues of the homogeneous boundary value problem. Note that the Green's function of part (b) fails to exist when $\lambda = 0$ or when $\lambda = n^2\pi^2$ where $n \in \mathbb{N}$. We have already shown that $\lambda = 0$ is *not* an eigenvalue; in this case the Green's function does exist, but it is only given by a formula different from the one determined in part (b). However, $\lambda = n^2\pi^2$ where $n \in \mathbb{N}$ is an eigenvalue as can be easily checked.
(3 points)